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REVIEW

A note on the paper "A new general eighth-order family of iterative methods for solving nonlinear equations"

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Abstract In this work, we briefly talk about the incorrect convergence order presented in the paper Khan et al. (Appl. Math. Lett. 25:2262–2266, 2012). Accordingly, we first show that their convergence theorem includes major errors, and it is attempted to provide the correct error equation of their method having fifth-order not eighth-order convergence. Finally, to support our assertion, some numerical examples are tested.

Keywords Nonlinear equations · Iterative methods · Multipoint methods · Convergence order

Convergence analysis

Khan et al. [1], proposed the following three-step iterative method

$$\begin{cases} y_m = x_m - \dfrac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - G\left(\dfrac{f(y_m)}{f(x_m)}\right)\dfrac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \dfrac{\mu}{\lambda + \nu q_m^2}\dfrac{f(z_m)}{K - C\,(y_m - z_m) - D\,(y_m - z_m)^2}, \end{cases} \tag{1}$$

where

$$\begin{cases} H = \dfrac{f(x_m) - f(y_m)}{x_m - y_m}, \\ K = \dfrac{f(y_m) - f(z_m)}{y_m - z_m}, \\ D = \dfrac{f'(x_m) - H}{(x_m - y_m)(x_m - z_m)} - \dfrac{H - K}{(x_m - z_m)^2}, \\ C = \dfrac{H - K}{(x_m - y_m)(x_m - z_m)} - D\,(x_m + y_m - 2z_m), \\ q_m = \dfrac{f(z_m)}{f(x_m)}, \end{cases} \tag{2}$$

and λ, μ, $\nu \in R$ and $G(t)$ represent a real-value function.

Khan et al. assert that if the conditions of the following theorem hold, then the iterative method (1.1) has convergence order eight (see Theorem in Section 4 in [1]). However, we prove that this is not true and prove that its convergence order is five.

Theorem 1.0.1 Let $f : D \subset R \to R$ have a single root $x^* \in D$, for an open interval D. If the initial point x_0 is sufficiently close to x^*, then the sequence $\{x_m\}$ generated by any method of the family (1.1) converges to x^*. If G is any function with $M_0 = G(0) = 1$, $M_1 = G'(0) = 2$, $M_2 = G''(0) < \infty$ and $\lambda = \mu \neq 0$, then the methods defined by (1.1) have convergence order of at least 5.

Proof For the sake of simplicity, we drop the iterative index m. Let $e = x - x^*$, and $c_k = \frac{f^{(k)}(x^*)}{k! f'(x^*)}$, $k = 0, 1, 2, \ldots$. Since $f(x^*) = 0 \neq f'(x^*)$, we can write

$$\begin{aligned} f(x) &= f'(x^*)\sum_{k=1}^{8} c_k e^k + O(e^9), \\ f'(x) &= f'(x^*)\sum_{k=1}^{7} k c_k e^{k-1} + O(e^8). \end{aligned} \tag{3}$$

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Let $e_y = y - x^*$. Then,

$$e_y = e - \frac{f(x)}{f'(x)} = c_2e^2 + \left(2c_3 - 2c_2^2\right)e^3 + \left(4c_2^3 - 7c_3c_2 + 3c_4\right)e^4 + \left(-8c_2^4 + 20c_3c_2^2 - 10c_4c_2 - 6c_3^2\right)e^5 + O(e^6). \quad (4)$$

Set,

$$t = \frac{f(y)}{f(x)} = c_2e + \left(2c_3 - 3c_2^2\right)e^2 + \left(8c_2^3 - 10c_3c_2 + 3c_4\right)e^3 \quad (5)$$

$$+ \left(-20c_2^4 + 37c_3c_2^2 - 14c_4c_2 - 8c_3^2\right)e^4 + \left(48c_2^5 - 118c_3c_2^3 + 51c_4c_2^2 + 55c_3^2c_2 - 22c_3c_4\right)e^5 + O(e^6). \quad (6)$$

Now, define the weight function $G(t)$ by

$$G(t) = M_0 + M_1t + M_2t2. \quad (7)$$

Then,

$$\begin{aligned} e_z = z - x^* &= e_y - G(t)\frac{f(y)}{f'(x)} - (c_2(M_0 - 1))e^2 \\ &+ \left(c_2^2(4M_0 - M_1 - 2) - 2c_3(M_0 - 1)\right)e^3 \\ &+ \left(c_2^3\left(-13M_0 + 7M_1 - \frac{M_2}{2} + 4\right) \right.\\ &\left. + c_3c_2(14M_0 - 4M_1 - 7) - 3c_4(M_0 - 1)\right)e^4 \\ &+ \left(c_3c_2^2(-64M_0 + 38M_1 - 3M_2 + 20)\right. \\ &+ 2c_4c_2(10M_0 - 3M_1 - 5) + 2c_3^2(6M_0 - 2M_1 - 3) \\ &\left.+ c_2^4(38M_0 - 33M_1 + 5M_2 - 8)\right)e^5 + O(e^6). \end{aligned} \quad (8)$$

If $M_0 = 1$ and $M_1 = 2$, then

$$e_z = \left(c_2^3\left(5 - \frac{M_2}{2}\right) - c_2c_3\right)e^4 + \left(c_2^4(5M_2 - 36) + c_3c_2^2(32 - 3M_2) - 2c_4c_2 - 2c_3^2\right)e^5 + O(e^6). \quad (9)$$

Now, assume that H, K D, and C are given by (1.2). Also, let $q = \frac{f(z)}{f(x)}$ and $\lambda = \mu \neq 0$. Consequently, the final error equation, i.e., $\hat{e}$, for the method (1.1) is obtained as follows

$$\hat{e} = e_z - \frac{\mu}{\lambda + vq^2}\frac{f(z)}{K - C(y - z) - D(y - z)^2} = \frac{1}{2}\left(c_2^5(M_2 - 10) + 2c_3c_2^3\right)e^5 + O(e^6), \quad (10)$$

which completes the proof. □

Table 1 Numerical results for $v = 0 = \omega$

Functions	$\lvert x_1 - x^*\rvert$	$\lvert x_2 - x^*\rvert$	$\lvert x_3 - x^*\rvert$	COC
$f_1(x)$	0.2606 (−5)	0.1547 (−31)	0.1141 (−162)	5.00
$f_2(x)$	0.60034 (−3)	0.3359 (−14)	0.1855 (−70)	4.99
$f_3(x)$	0.4018 (−2)	0.2348 (−12)	0.1509 (−63)	5.00

Numerical performances

This section concerns with numerical results of the proposed methods (1.1). We take the derived method from it by considering $\lambda = \mu = 1$ and $G(t) = \frac{1}{1-2t+\omega t^2}$, where $\omega \in R$. This is the Method 1 in [1] (see Equations 21 and 22 in Section 5 there.)

$$\begin{cases} y_m = x_m - \dfrac{f(x_m)}{f'(x_m)}, \\ z_m = y_m - \dfrac{f^2(x_m)}{f^2(x_m) - 2f(x_m)f(y_m) + \omega f^2(y_m)}\dfrac{f(y_m)}{f'(x_m)}, \\ x_{m+1} = z_m - \dfrac{1}{1 + vq_m^2}\dfrac{f(z_m)}{K - C(y_m - z_m) - D(y_m - z_m)^2}, \end{cases} \quad (11)$$

where

$$\begin{cases} H = \dfrac{f(x_m) - f(y_m)}{x_m - y_m}, \\ K = \dfrac{f(y_m) - f(z_m)}{y_m - z_m}, \\ D = \dfrac{f'(x_m) - H}{(x_m - y_m)(x_m - z_m)} - \dfrac{H - K}{(x_m - z_m)^2}, \\ C = \dfrac{H - K}{(x_m - y_m)(x_m - z_m)} - D(x_m + y_m - 2z_m), \\ q_m = \dfrac{f(z_m)}{f(x_m)}. \end{cases} \quad (12)$$

Numerical results have been carried out using Mathematica 9 with 200 digits of precision. $a(-b)$ means $a \times 10^b$. In each table, COC stands for computational order of convergence (see [1]) which is given by

$$rho \approx \frac{\ln(\lvert x_{m+1} - x_m\rvert \lvert x_m - x_{m-1}\rvert^{-1})}{\ln(\lvert x_m - x_{m-1}\rvert \lvert x_{m-1} - x_{m-2}\rvert^{-1})}.$$

Among many test problems, the following four examples are considered

$$f_1(x) = x^2 - 9, \quad x^* = 3,\ x_0 = 2.6$$

$$f_2(x) = (x - 2)(x^6 + x^3 + 1)e^{-x^2}, \quad x^* = 2,\ x_0 = 1.8,$$

$$f_3(x) = \prod_{k=1}^{12}(x - k), \quad x^* = 5,\ x_0 = 5.3.$$

To sum up, it can be concluded the that the method (1.1) has fifth-order convergence. Therefore, authors' claim is not true that they have presented a family of iterative methods for solving nonlinear equations with eighth-order convergence.

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Reference

1. Khan, Y., Fardi, M., Sayevand, K.: A new general eighth-order family of iterative methods for solving nonlinear equations. Appl. Math. Lett. **25**, 2262–2266 (2012)